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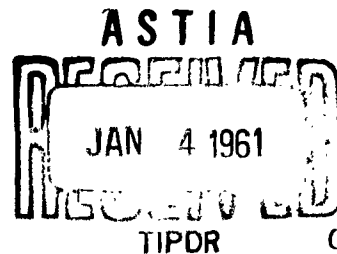# UNCLASSIFIED

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Density Fluctuations in a Plasma In a Magnetic Field, With Applications to the Ionosphere

by
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ASTIA

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RADIOSCIENCE LABORATORY

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DENSITY FLUCTUATIONS IN A PLASMA IN A MAGNETIC FIELD, WITH APPLICATIONS TO THE IONOSPHERE

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ABSTRACT

General expressions are developed for the fluctuation in density
of electrons, ions, and charge in a plasma in thermal equilibrium in
an external magnetic field taking only Coulomb interaction into account.
The spectral distribution of the spatial Fourier components of these
fluctuations is derived from basic principles.

The fluctuations in electron density are discussed in some detail,
and spectra are computed under conditions which are thought to prevail
in the outer ionosphere. Frequency spectra of general validity are com-
puted for electron density fluctuations along the magnetic field. It
is shown by means of examples that the frequency spectra under ionospheric
conditions are not much influenced by the magnetic field except for den-
sity fluctuations fairly close to perpendicularity to the magnetic field.

Applications to incoherent backscattering are discussed, and it is
shown that, under suitable conditions, backscatter techniques can give
valuable information about electron density, temperature and consti-
tuents of the ionosphere.

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I. INTRODUCTION

Much of the current interest in the fluctuation of electron density in a plasma in thermodynamic equilibrium stems from an original suggestion by W. E. Gordon (1958) that such fluctuations might be detectable in the upper ionosphere in backscatter observations with powerful radar systems. Gordon predicted the total scattering cross section by adding up the cross sections of the individual electrons, and he estimated the spectral broadening by ascribing to the scattering elements the thermal velocity of individual electrons.

When Bowles observed this type of scattering (1958, 1959) it appeared that the total power was of the right order of magnitude, but that the spectrum was nowhere near as wide as expected. This has been confirmed by recent observations by Pineo, et al, (1960). Subsequently several authors have described theories to explain this discrepancy and have come up with essentially similar results (Fejer 1960, Salpeter 1960a, 1960b, Dougherty and Farley 1960a, 1960b). The reason for the narrow spectrum is the interaction between the electrons and the ions. This interaction causes the spectral width to be that which would have resulted if the electrons had had the mass of the ions and had been completely free to move, very roughly stated. The theories referred to show that the actual spectral distribution is considerably more complicated if all the details are taken into account. Only Dougherty and Farley have taken the effect of the geomagnetic field properly into account.

The present work is concerned with the calculation of the electron density fluctuations in the general case of ions with an arbitrary number of positive charges and in the presence of an external magnetic field. The effect of two-body collisions is taken approximately into account by introducing a relaxation term in the Boltzmann transfer equation. The derivation does not depend on Nyquist's theorem as does that of Dougherty and Farley, but their result is obtained in the case of singly charged ions and no collisions. When the magnetic field is neglected, but the multiple ion charge retained, the result of Salpeter (1960b) is recovered.

The calculation of the total fluctuation is first carried out along lines corresponding to those used by Pines and Bohm (1952). These total fluctuations are shown to be independent of an external magnetic field and independent of two-body collisions. Time variations of the density fluctuations---or the spectrum---are then discussed by means of the Boltzmann equation. The solution of this equation follows very closely that of Bernstein (1958) who investigated waves in a plasma in a magnetic field. We essentially have only to supply the appropriate initial conditions to the solutions given by Bernstein. It turns out that the fluctuations in ion density and charge density and their spectral distribution may be obtained with very little extra work. These results are therefore given in addition to the electron density spectra. They are of no direct interest to the radar back-scatter observations, but may be of use in other problems, such as determining friction and diffusion coefficients of the Fokker-Planck equation for the plasma (Hubbard 1960).

Numerical calculations of the spectral distribution of the electron density fluctuations are made for a set of parameters covering a large range of temperatures, plasma frequencies and radar frequencies appropriate to ionospheric conditions. The results are given as sets of curves which can be conveniently used for interpreting backscatter observations.

The extension of the theory to a larger number of ionic constituents is straightforward in principle, but is not carried out here.

II. RELATIONS BETWEEN PLASMA FLUCTUATIONS AND SCATTERING PROPERTIES

Before we start discussing plasma properties we must relate the scattered energy to the electron density fluctuations. For weak scattering, i.e., when the Born approximation can be used, one obtains for the scattering cross section per unit solid angle, per unit incident power density and per unit scattering volume

$$\sigma = \sigma_e V \; \langle |n(\vec{k})|^2 \rangle_{av} \tag{1}$$

where σ_e is the scattering cross section per unit solid angle and per unit incident power density of a single electron, and where $\vec{K}$ is the difference between the wave vectors of the incident and the scattered wave. Note that σ_e depends both on the polarization and the wave vector difference $\vec{K}$. The quantity $n(\vec{k})$ is defined as

$$n(\vec{k}) = \frac{1}{V} \int_V d(\vec{r})\ n(\vec{r})\ \exp(i\vec{k}\vec{r})$$

where $n(\vec{r})$ is the number density of electrons. If, in addition, one requires the power spectrum of the scattered energy, one must study the scattering cross section

$$\sigma_\omega = \sigma_e\ V < |n(\vec{k},\omega)|^2 >_{av} \tag{2}$$

where ω is the difference of the angular frequency of the incident wave and the scattered power under consideration. The quantity $<|n(\vec{k},\omega)|^2>_{av}$ is simply the power spectrum of the spatial Fourier component of wave vector $\vec{k}$. The two scattering cross sections are related through the equation

$$\sigma = \int_{-\infty}^{+\infty} \sigma_\omega\ d\omega \tag{3}$$

Having now established the relationships between the scattering properties of the plasma and the density fluctuations we can turn to the study of the latter.

III. TOTAL FLUCTUATIONS IN THE PLASMA

Assume that the plasma is neutral and that the average number density of positive ions is N_o and the average number density of electrons is n_o. The number of charges on the ions is $Z = n_o/N_o$. The densities may be expressed as

$$N(\vec{r}) = \sum_{i=1}^{N_o V} \delta(\vec{r} - \vec{R}_i)$$

and
$$n(\vec{r}) = \sum_{i=1}^{n_o V} \delta(\vec{r} - \vec{r}_i)$$
(4)

Here V is the volume of a large periodicity cube with sides L, and $\vec{R}_i$ and $\vec{r}_i$ are the positions of the ions and the electrons. The charge density then becomes

$$\rho(\vec{r}) = e \ [ZN(\vec{r}) - n(\vec{r})]$$
(5)

and the spatial Fourier component becomes

$$\rho(\vec{k}) = e \ [ZN(\vec{k}) - n(\vec{k})]$$
(6)

with
$$N(\vec{k}) = \frac{1}{V} \int_V d(\vec{r}) \ N(\vec{r}) \ \exp(i\vec{k}\vec{r})$$

and with a similar expression for $n(\vec{k})$. The wave vector $\vec{k}$ is given by $\vec{k} = 2\pi(\ell_1, \ell_2, \ell_3)/L$ where the ℓ's are integers between $-\infty$ and $+\infty$.

The interaction between the various charged particles may now be taken into account through an electric field $\vec{E}(\vec{r})$ which may also be expanded within the periodicity cube $V = L^3$ in a Fourier series. By far the most important interaction in a nonrelativistic plasma is through the Coulomb forces. This means that we let the velocity of interaction become infinite so that the electric field may be derived from a scalar potential which can be found from Poisson's equation. We therefore obtain

$$\vec{E}(\vec{k}) = \frac{i\vec{k}}{\epsilon_o k^2} \ \rho(\vec{k})$$
(8)

with ϵ_o = dielectric constant in vacuo. This will be a good approximation as long as the thermal energy of the electrons is considerably smaller than the relativistic rest energy of the electrons, i.e.,

$$\frac{KT}{mc^2} \ll 1.$$

The total energy of the plasma, including the self energy of the individual particles and their kinetic energies, then becomes

$$W = \frac{1}{2} \sum_{i=1}^{N_o V} MV_i^2 + \frac{1}{2} \sum_{i=1}^{n_o V} mv_i^2 + \frac{1}{2} \epsilon_o \int_V |\vec{E}(\vec{r})|^2 \, d(\vec{r})$$

(9)

where M is the mass of an ion and m that of an electron. Using Parceval's theorem, the last part of this may be expressed in terms of Fourier components as follows:

$$W_{el.st.} = \frac{1}{2} \epsilon_o V \sum_{\vec{k}} |\vec{E}(\vec{k})|^2 = \frac{1}{2} \frac{V}{\epsilon_o} \sum_{\vec{k}} \frac{|\rho(\vec{k})|^2}{k^2} =$$

$$= \frac{Ve^2}{2\epsilon_o} \sum_{\vec{k}} \frac{1}{k^2} |[ZN(\vec{k}) - n(\vec{k})]|^2$$

(10)

We note in particular that this expression will remain the same whether there is an external magnetic field present or not, and it will not be altered by the presence of neutral particles with which the ions and the electrons may collide.

If $N(\vec{r})$ and $n(\vec{r})$ were continuous functions of position, we know from information theory that in order to determine the Fourier components up to $\vec{k} = 2\pi(\ell_1, \ell_2, \ell_3)/L$ the number of sampling points required in space is very close to $8(\ell_1 \ell_2 \ell_3)$ (Brillouin 1956). If, therefore, $\ell_1 \ell_2 \ell_3 \ll n_o V$ and $N_o V$, it means that many particles must contribute to each sampled value. Let these sampled values be denoted by $N_1 \ldots N_{8(\ell_1 \ell_2 \ell_3)}$ for ions and $\bar{n}_1 \ldots \bar{n}_{8(\ell_1 \ell_2 \ell_3)}$ for electrons.

It is of interest to know how these sampled values, or occupation numbers, are related to the actual discontinuous functions $N(\vec{r})$ and $n(\vec{r})$. Consider wave numbers $\vec{k}_{n_1,n_2,n_3}$ where $|n_1|$, $|n_2|$, and $|n_3|$ are smaller than or equal to ℓ_1, ℓ_2 and ℓ_3, respectively. The number of sampling points required along the three axes is then $2\ell_1 + 1$, $2\ell_2 + 1$ and $2\ell_3 + 1$. Again, from information theory, it follows that the sampled values (occupation numbers) may be obtained from $N(\vec{r})$ and $n(\vec{r})$ by integration over the periodicity cube with the following weighting factor:

$$
f(\vec{r}-\vec{r}_{m_1,m_2,m_3}) = \prod_{i=1}^{3} \frac{\sin\left[\frac{2\ell_1+1}{L}\pi\left(x_1 - \frac{m_1 L}{2\ell_1+1}\right)\right]}{(2\ell_1+1)\sin\left[\frac{\pi}{L}\left(x_1 - \frac{m_1 L}{2\ell_1+1}\right)\right]} \tag{11}
$$

If we denote the sampled values at $\vec{r}_{m_1,m_2,m_3} = L\left[\frac{m_1}{2\ell_1+1}, \frac{m_2}{2\ell_2+1}, \frac{m_3}{2\ell_3+1}\right]$

by $\bar{\bar{N}}(\vec{r}_{m_1,m_2,m_3})$ and $\bar{n}(\vec{r}_{m_1,m_2,m_3})$, it can be shown that

$$
\bar{N}(\vec{r}_{m_1,m_2,m_3}) = \frac{V}{\prod_{i=1}^{3}(2\ell_1+1)} \sum_{n_1=-\ell_1}^{\ell_1} \sum_{n_2=-\ell_2}^{\ell_2} \sum_{n_3=-\ell_3}^{\ell_3} N(\vec{k}_{n_1,n_2,n_3}) \cdot
$$

$$
\cdot \exp(-i\vec{k}_{n_1,n_2,n_3}\,\vec{r}_{m_1,m_2,m_3}) \tag{12}
$$

The complete set of sampled values (occupation numbers) is therefore fully determined by the spectral components in $\vec{k}_{n_1,n_2,n_3}$ with n_1,n_2,n_3 ranging in magnitude from zero to ℓ_1,ℓ_2,ℓ_3. It follows that

$$
\sum_{m_1,m_2,m_3} \bar{N}^2(r_{m_1,m_2,m_3}) = \frac{V^2}{8(\ell_1\ell_2\ell_3)} \sum_{n_1,n_2,n_3} |N(\vec{k}_{n_1,n_2,n_3})|^2
$$

$$
\tag{13}
$$

3.1 THE DISTRIBUTION OF THE SAMPLED VALUES

We now consider the thermodynamic subsystem consisting of the occupation number $\bar{N}_1 \ldots \bar{N}_{8(\ell_1\ell_2\ell_3)}$ and $\bar{n}_1 \ldots \bar{n}_{8(\ell_1\ell_2\ell_3)}$. Assuming that the velocities of the individual particles are statistically unrelated to these occupation numbers, we conclude that the probability of one particular of the states corresponding to a sequence of occupation numbers is given by a Gibbs distribution function:

$$\exp\left[-W(..\bar{N}_1..,..\bar{n}_1..)/KT\right] \qquad (14)$$

The permutability of these states is given by

$$\frac{(N_0 V)!}{\bar{N}_1!\ldots\ldots\bar{N}_{8(\ell_1\ell_2\ell_3)}!} \cdot \frac{(n_0 V)!}{\bar{n}_1!\ldots\ldots\bar{n}_{8(\ell_1\ell_2\ell_3)}!} \qquad (15)$$

The probability density of a sequence of occupation numbers therefore can be expressed as

$$p(.\bar{n}_j.,\bar{N}_j.) \sim \frac{(N_0 V)!}{\prod_i \bar{N}_i!} \; \frac{(n_0 V)!}{\prod_i \bar{n}_i!} \exp(-W/KT) \qquad (16)$$

As long as the occupation numbers are fairly large we can use Stirling's formula for the factorials. The distribution then simplifies to

$$p \sim \exp(-W/KT) \cdot \exp\left[-\frac{8(\ell_1\ell_2\ell_3)}{2n_0 V}\sum_i\left(\bar{n}_i^2 + \frac{N_0}{n_0}\bar{N}_i^2\right)\right]$$

$$(17)$$

From Eq. (13) we know how the occupation numbers may be expressed in terms of Fourier components. We must note, however, that Eq. (13) has twice as many terms on the right-hand side as on the left because $N(\vec{k})$ has both real and imaginary parts. In changing the variables from occupation numbers to Fourier components we therefore count only directions of $\vec{k}$ pointing into one hemisphere provided we want to use n_1,

N_i, n_r and N_r as independent variables. In this case N_i and N_r stand for the imaginary and the real parts of $N(\vec{k})$, with similar notation for the electrons. Because the Fourier components are linearly related to the occupation numbers (see Eq. 12), the Jacobian of the transformation is a constant. We therefore obtain for the joint distribution of the real and imagainary parts of the Fourier components

$$p(..N_r,n_r,N_i,n_i,..) \sim \exp\left(-\frac{V}{n_o}\sum_{\substack{n_1=n_2=n_3=\\0\ -\ell_2\ -\ell_3}}^{\ell_1\ \ell_2\ \ell_3}\{2x_p^2\ [z^2(N_r^2 + N_i^2) +\right.$$

$$\left. + (n_r^2 + n_i^2) - 2Z(N_r n_r + N_i n_i)] + (ZN_r^2 + n_r^2 + ZN_i^2 + n_i^2)\}\right)$$

(19)

Here we have introduced $(2x_p^2)^{-1} = (D|\vec{k}_{n_1,n_2,n_3}|)^2$ with

$$D^2 = \epsilon_o KT/n_o e^2$$

D is known as the Debye length (Spitzer 1956).

We immediately see that this is a multidimensional probability density of the Gaussian type. We note that the different Fourier components enter through products of distribution functions for each. Therefore, we conclude that the Fourier components corresponding to different wave numbers are statistically independent. In each of the elementary distribution functions we see that even the real and imaginary parts corresponding to the same wave number are independent. We can therefore write down the expression for the distribution of the real parts of $N(\vec{k})$ and $n(\vec{k})$ for one particular wave number eseparately:

$$p(N_r,n_r) \sim \exp\ \{-\frac{V}{n_o}\ [N_r^2 Z(1 + 2x_p^2 Z) + n_r^2(1 + 2x_p^2)$$

$$- 4Zx_p^2 N_r n_r]\}$$

(20)

Comparing this with the standard form of a two-dimensional Gaussian distribution we conclude that

$$\langle N_r^2 \rangle_{av} = \langle N_i^2 \rangle_{av} = \frac{n_o}{2VZ} \frac{2X_p^2 + 1}{1 + 2X_p^2(1 + Z)}$$

$$\langle n_r^2 \rangle_{av} = \langle n_i^2 \rangle_{av} = \frac{n_o}{2V} \cdot \frac{Z2X_p^2 + 1}{1 + 2X_p^2(1 + Z)}$$

$$\langle n_r N_r \rangle_{av} = \langle n_i N_i \rangle_{av} = \frac{n_o}{2V} \frac{2X_p^2}{1 + 2X_p^2(1 + Z)} \tag{21}$$

and

$$\langle n_i N_r \rangle_{av} = \langle n_r N_i \rangle_{av} = 0$$

3.2 DISCUSSION OF RESULTS

The fluctuation in electron density becomes

$$\langle |n(\vec{k})|^2 \rangle_{av} = \frac{n_o}{V} \frac{1 + 2X_p^2 Z}{1 + 2X_p^2(1 + Z)} \tag{22}$$

and the fluctuation in ion density

$$\langle |N(\vec{k})|^2 \rangle_{av} = \frac{n_o}{ZV} \frac{1 + 2X_p^2}{1 + 2X_p^2(1 + Z)} \tag{23}$$

We note that when $Z = 1$ the number density fluctuations in ion and electron densities become identical. For small values of $|\vec{k}|$ the fluctuations are 1/2 the value they would have been in a gas without particle interaction. At large wave numbers the fluctuations become identical with those in a gas with no particle interaction. The above results agree in every detail with those of Salpeter (1960a).

Because we know the correlation between the motion of the ions and the electrons we can also write down the expression for the mean potential energy associated with each wave number:

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$$V_k = \frac{1}{2}\epsilon_o V \, |\vec{E}(\vec{k})|^2 = \frac{Ve^2}{2\epsilon_o k^2} \, |ZN(\vec{k}) - n(\vec{k})|^2 =$$

$$= \frac{1}{2} KT \frac{2X_p^2(1+Z)}{1 + 2X_p^2(1+Z)} \qquad (24)$$

This reduces to the result of Pines and Bohm (1952) if we put $Z = 0$, i.e., if we let the ions become smeared out to a uniform background of positive charge.

We note that the above results are independent of the presence of a magnetic field or two-body collisions (see Eq. 10). This has only been shown here for the case of a plasma with Coulomb interaction. Similar conclusions have been reached by Dougherty and Farley through entirely different considerations.

IV. THE SPECTRAL DISTRIBUTION OF THE SPATIAL COMPONENTS

We next desire to know how the density fluctuations vary with time. A relationship describing these time variations is given by the Boltzmann transfer equation. This equation has been solved for the case of a plasma in an external magnetic field (Gross 1951, Gordeyev 1952 and Bernstein 1958). It is, therefore, here necessary only to take over these results and apply them to our problem directly. We do have to supply some sort of initial conditions to obtain the answer we want. As we shall see, these can be found from the results of the previous section. For completeness the method for solving the Boltzmann equation is briefly outlined.

4.1 SOLUTION OF THE BOLTZMANN EQUATION

The Boltzmann equation for the particle density in phase space is given by:

$$\frac{\partial n(\vec{r},\vec{v},t)}{\partial t} + \vec{v}\,\frac{\partial n(\vec{r},\vec{v},t)}{\partial \vec{r}} + \mu\,[\vec{E}(\vec{r},t) + \vec{v}\times\vec{B}(\vec{r},t)]\,\frac{\partial n(\vec{r},\vec{v},t)}{\partial \vec{v}} = \left(\frac{\delta n}{\delta t}\right)_{coll}$$

$$(25)$$

Here $\mu = -e/m$ for electrons and Ze/M for the ions. Assuming that the deviations from a maxwellian velocity distribution are fairly small, we put

$$n(\vec{r},\vec{v},t) = n_o(\vec{v})\,[1 + n_1(\vec{r},\vec{v},t)] \qquad (26)$$

with $n_o(\vec{v})$ a maxwellian and n_1 small compared with unity. Again, introducing spatial Fourier transforms by

$$n_1(\vec{r},\vec{v},t) = \sum_{\vec{k}} n_1(\vec{k},\vec{v},t)\,\exp(-i\vec{k}\vec{r}) \qquad (27)$$

and Laplace transforms through

$$n_1(\vec{k},\vec{v},s) = \int_0^\infty n_1(\vec{k},\vec{v},t)\,\exp(-st)\,dt \qquad (28)$$

the linearized equation becomes

$$s n_1(\vec{k},\vec{v},s) - n^1(\vec{k},\vec{v}) - i\vec{k}\vec{v}n_1(\vec{k},\vec{v},s) + \mu\left[\frac{1}{n_o(\vec{v})}\frac{\partial n_o}{\partial \vec{v}}\,\vec{E}(\vec{k},s) - \vec{B}(\vec{v}X\frac{\partial n_1}{\partial \vec{v}})\right] =$$

$$= -\,\nu n_1(\vec{k},\vec{v},s) \qquad (29)$$

The quantity $n^1(\vec{k},\vec{v})$ is the initial value obtained by putting $t = 0$ in $n_1(\vec{k},\vec{v},t)$. In the above, the collisions have been taken into account by means of a relaxation term. The quantity ν is an effective collision frequency. By combining the relaxation term with the first term on the left-hand side we obtain

$$s n_1(\vec{k},\vec{v},s) \rightarrow (s + \nu)\,n_1(\vec{k},\vec{v},s) = s'n_1(\vec{k},\vec{v},s)$$

We can therefore neglect the collisions in the calculations provided we remember that wherever s appears explicitly it actually stands for the sum of s and the collision frequency.

To solve the differential equation in $\vec{v}$ one may introduce cylindrical coordinates with axis along the magnetic field and with propagation

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vector in the 1-3 plane, as shown in Fig. 1. With these w coordinates Eq. (29) becomes

$$\frac{\partial n_1}{\partial \phi} - \frac{1}{\mu B} \left[s - ik(u \cos \theta + w \sin \theta \cos \phi) \right] n_1 =$$

$$= \frac{1}{\mu B} \left[\frac{\mu}{n_0(\vec{v})} \frac{\partial n_0}{\partial \vec{v}} \vec{E} - n^1(\vec{k},\vec{v}) \right] \tag{30}$$

The integration of Eq. (30), which has been performed by Bernstein (1958), is briefly outlined in Appendix A.

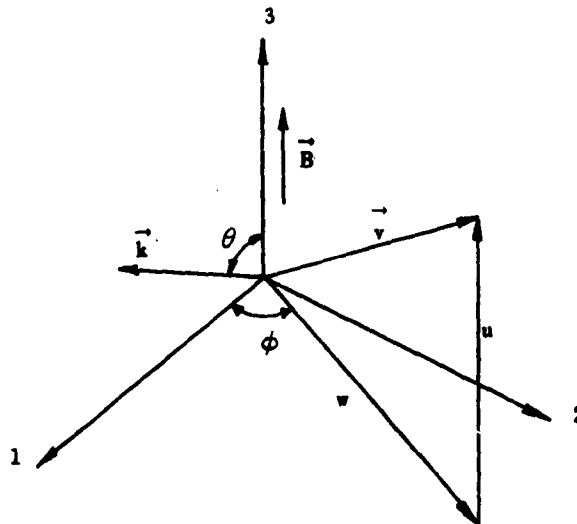FIG. 1.   COORDINATE SYSTEM FOR SOLVING EQ. (29)

The solution is:

$$n_1(\vec{k},\vec{v},s) = \frac{1}{\mu B} \int_{\infty(\text{sign})\mu}^{\phi} \frac{\mu}{n_0} \frac{\partial n_0'}{\partial \vec{v}} \vec{E}(\vec{k},s) - n^1(\vec{k},\vec{v}') \cdot d\phi'$$

$$\exp \left\{ \frac{1}{\mu B} \left[(s - iku \cos \theta)(\phi - \phi') - ikw \sin \theta (\sin \phi - \sin \phi') \right] \right\} \tag{31}$$

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If this is written out explicitly for electrons and ions, one obtains

$$n_1(\vec{k},\vec{v},s) = -\int_{-\infty}^{\emptyset} G_e(\emptyset,\emptyset') \left\{ n^1(\vec{k},\vec{v'}) - \frac{21X_p^2}{n_o} \vec{k}\vec{v'} [ZN(\vec{k},s)-n(\vec{k},s)] \right\} d\emptyset'$$

(32)

and

$$N_1(\vec{k},\vec{V},s) = -\int_{-\infty}^{\emptyset} G_i(\emptyset,\emptyset') \left\{ N^1(\vec{k},\vec{V'}) + Z\frac{21X_p^2}{n_o} \vec{k}\vec{V'} [ZN(\vec{k},s)-n(\vec{k},s)] \right\} d\emptyset'$$

(33)

where the integrating factors G_e and G_i follow from Eq. (31).

We next remove fluctuations in velocity space by averaging over all velocities by means of the maxwellian distribution, that is, we form

$$n(\vec{k},s) = \int n_o(\vec{v}) \; n_1(\vec{k},\vec{v},s) \; d(\vec{v})$$

$$N(\vec{k},s) = \int N_o(\vec{V}) \; N_1(\vec{k},\vec{V},s) \; d(\vec{V})$$

(34)

This leads to the equations

$$n(\vec{k},s) = Y_e(\vec{k},s) - \frac{1}{n_o} 2X_p^2 R_e \{ZN(\vec{k},s) - n(\vec{k},s)\}$$

$$N(\vec{k},s) = Y_i(\vec{k},s) + \frac{1}{n_o} 2X_p^2 R_i \{ZN(\vec{k},s) - n(\vec{k},s)\}$$

(35)

The complete expressions for Y and R introduced both for ions and for electrons are given in Appendix B. Solving the set of equations (35) one obtains

$$n(\vec{k},s) = \frac{Y_e(1 - \frac{1}{n_o} 2X_p^2 Z^2 R_i) - Y_i\frac{1}{n_o} 2X_p^2 ZR_e}{1 - \frac{1}{n_o} 2X_p^2(R_e+Z^2R_i)}$$

(36)

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$$N(\vec{k},s) = \frac{Y_i\left(1 - \frac{1}{n_0} 2X_p^2 R_e\right) - Y_e \frac{1}{n_0} 2X_p^2 Z R_i}{1 - \frac{1}{n_0} 2X_p^2 (R_e + Z^2 R_i)} \qquad (37)$$

From the above two equations the variation of electron density and ion density with time can be found by taking the inverse Laplace transform. Only the time variation of the electron density will be computed in detail here, as the procedure for dealing with the ion density variations is similar.

For the electron density the inverse transform becomes

$$n(\vec{k},t) = \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} n(\vec{k},s) \exp(st) \, ds \qquad (38)$$

The electron density at time t can therefore be found, provided the initial conditions at time $t = 0$ are known. Because of the statistical nature of the problem, however, we cannot fix the initial conditions. We must therefore resort to a statistical description. Hence we form the following expression

$$\langle n^*(\vec{k},0)n(\vec{k},t)\rangle_{av} = \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \langle n^*(\vec{k},0)\, n(\vec{k},s)\rangle_{av} \exp(st) \, ds$$

$$(39)$$

The spectrum is the Fourier transform of an autocorrelation function, defined by (Landau and Lifshitz 1958)

$$\frac{1}{2} \langle n^*(\vec{k},0)\, n(\vec{k},t) + n(\vec{k},0)\, n^*(\vec{k},t)\rangle_{av} \qquad (40)$$

Because of the symmetry properties of $n(\vec{k},t)$ this turns out to be equal to

$$2\mathrm{Re}\langle n^*(\vec{k},0)n(\vec{k},t)\rangle_{av}$$

By means of this expression and by using the Wiener-Khinchine theorem, we conclude that the spectral distribution is given by

$$\langle |n(\vec{k},\omega)|^2 \rangle_{av} = \frac{1}{\pi} \lim_{Re(s)\to 0} Re \ \langle n*(\vec{k},0) \ n(\vec{k},s) \rangle_{av} \qquad (41)$$

Equation (41) is the formal solution of our problem. In order to be able to compute the actual spectra we must study in some detail the function $\langle n*(\vec{k},0)n(\vec{k},s)\rangle_{av}$ which is given by

$$\frac{\langle n*(\vec{k},0)Y_e \rangle_{av} (1 - \frac{1}{n_o} 2X_p^2 Z^2 R_i) - \langle n*(\vec{k},0)Y_i \rangle_{av} \frac{1}{n_o} 2X_p^2 ZR_e}{1 - \frac{1}{n_o} 2X_p^2 (R_e + Z^2 R_i)}$$

$$(42)$$

In particular, the expressions for the averages contain terms of the form

$$\langle n*(\vec{k},0)n^1(\vec{k},\vec{v})\rangle_{av}$$

and

$$\langle n*(\vec{k},0)N^1(\vec{k},\vec{V})\rangle_{av} \qquad (43)$$

In the previous section we assumed that the spatial density fluctuations are independent of the velocities of the individual particles. As long as this is true, we conclude that the expressions (43) are both independent of velocity. Using Eqs. (21) and (22) we obtain

$$\langle n*(\vec{k},0)n^1(\vec{k},\vec{v})\rangle_{av} = \frac{1}{n_o} \langle |n(\vec{k})|^2 \rangle_{av} = \frac{1}{V} \frac{1+2X_p^2 Z}{1 + 2X_p^2(1+Z)}$$

$$\langle n*(\vec{k},0)N^1(\vec{k},\vec{V})\rangle_{av} = \frac{Z}{n_o} \langle n*(\vec{k})N(\vec{k})\rangle_{av} = \frac{1}{V} \frac{2ZX_p^2}{1 + 2X_p^2(1+Z)}$$

$$(44)$$

Substitution of these into expressions (42) and (41), together with the results of Appendix B we finally obtain for the spectrum

$$\langle |n(\vec{k},\omega)|^2 \rangle_{av} = \frac{n_o}{\pi V \omega} \frac{\text{Im}(-F_e)|1+2X_p^2 Z F_i|^2 + 4X_p^4 Z \text{Im}(-F_i)|F_e|^2}{|1+2X_p^2(F_e + Z F_i)|^2}$$

(45)

The function F, as shown in Appendix B, is defined in the following manner for electrons:

$$F_e = 1 - \left(i \frac{X}{X_e} + \Lambda_e\right)\int_0^\infty \exp\left\{-iy(\frac{X}{X_e}) - y\Lambda_e - \frac{1}{2X_e^2}[\sin^2\theta(1-\cos y) + \frac{1}{2}y^2\cos^2\theta]\right\} dy$$

Here we have defined the quantities X, X_e and Λ_e by

$$X^2 = \frac{m}{2KT} \frac{\omega^2}{k^2} \qquad X_e^2 = \frac{m}{2KT} \frac{\Omega_e^2}{k^2}$$

and

$$\Lambda_e = \frac{\nu_e}{\Omega_e}$$

Ω_e is the gyrofrequency of the electrons, eB/m, and ν_e is the effective collision frequency of the electrons. See Eq. (29) and comments. The expression for F_i may be obtained by substituting κX for X and $\kappa^{-1}X_e$ for X_e in the expression for F_e. κ is the square root of the ratio of ion and electron masses. One must also substitute the appropriate collision frequency for ions.

The corresponding expression for the ion density fluctuations become

$$\langle |N(\vec{k},\omega)|^2 \rangle_{av} = \frac{n_o}{\pi Z V \omega} \frac{\text{Im}(-F_i)|1+2X_p^2 F_e|^2 + \text{Im}(-F_e)4X_p^4 Z|F_i|^2}{|1+2X_p^2(F_e + Z F_i)|^2}$$

(46)

and for the charge density variations

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$$\langle|\rho(\mathbf{k},\omega)|^2\rangle_{av} = \frac{n_0 e^2}{\pi V \omega} \frac{\mathrm{Im}(-F_e) + Z\,\mathrm{Im}(-F_i)}{|1 + 2X_p^2(F_e + ZF_i)|^2} \qquad (47)$$

In the next section we go on to discuss the properties of the electron density spectra under conditions of interest in ionospheric applications.

V. DETERMINATION OF ELECTRON DENSITY SPECTRA

Before starting to discuss the shape of the spectra of electron density fluctuations for application to scattering in the ionosphere, we must determine from ionospheric data the ranges of the various parameters. It turns out that the spectra are quite straightforward to compute when $\vec{k}$ is parallel to the magnetic field. This longitudinal case is identical with the nonmagnetic case. When $\vec{k}$ is no longer parallel to the magnetic field, it is still possible to work out exact expressions for both F_e and F_i but these involve slowly convering infinite sums with terms proportional to Bessel functions of imaginary arguments and are, in general, difficult to handle. However, in the limiting cases of radius of gyration of the charged particle small or large in comparison with k^{-1}, more tractable approximate expressions can be used. In the ionosphere, within a fairly wide radio frequency range, the electrons will have a small radius of gyration, and the ions a large radius of gyration, compared with k^{-1}.

For strictly transverse propagation the spectrum cannot be worked out without retaining some sort of damping. As the extent of this damping is not known, the spectra can be discussed only qualitatively in this case.

5.1 RANGE OF THE PARAMETERS

We shall assume that over the height range of interest at present, say from about 300 km to roughly 2000 km, the temperature is of the order of $1000°$ to $2000°$K (Chapman, 1960). In the actual calculations

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of the spectra we shall consider only the case of thermal equilibrium between ions and electrons. As a representative value for the gyro-frequency of the electrons we shall take 1 Mc/s. The type of ions present is not at all well known. At greater heights there is some reason to believe that protons may predominate, whereas at lower heights oxygen, nitrogen and others may be present. To get some insight into the effect of the mass of the ions, spectra are given both for protons and for oxygen ions.

The plasma frequencies are taken to range from 0.5 to 5.0 Mc/s and the radio frequencies are assumed to lie in the range from 50 to 3000 Mc/s. One then obtains:

Range of $\qquad X_p^2 = \dfrac{m}{2KT} \left(\dfrac{\omega_p}{k}\right)^2 \qquad : \qquad 10^{-2} \rightarrow 7.10^5$

Range of $\qquad X_e^2 = \dfrac{m}{2KT} \left(\dfrac{\Omega_e}{k}\right)^2 \qquad : \qquad 4.10^{-2} \rightarrow 3.10^2$

Only singly ionized ions will be considered. The values for $\kappa = (M/m)^{\frac{1}{2}}$ are:

$$\begin{aligned} \text{Hydrogen} \quad &: \quad \kappa = 43 \\ \text{Oxygen} \quad &: \quad \kappa = 172 \end{aligned}$$

5.2 LONGITUDINAL CASE

This case is obtained by putting $\theta = 0$ in the integrals for F_e and F_i . First we would like to know whether collisions have to be taken into account. By studying the integrals we conclude that collisions are unimportant provided:

$$X_e \Lambda_e = X_{ce} \ll 1$$

and

$$X_i \Lambda_i = X_{ci} \ll 1$$

These conditions amount to assuming that the mean free paths of the electrons and the ions are larger than the scale of the spatial Fourier

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component under consideration, i.e., larger than k^{-1}. Since the mean free path at 200 km is of the order of several hundred metres, and because the wavelength of the radio waves will be at most a few metres, we conclude that collisions are unimportant. With no collisions we find:

$$\text{Re}(F_e) = 1 - 2X \exp(-X^2) \int_0^X \exp(\phi^2) d\phi$$

$$\text{Im}(-F_e) = \pi^{\frac{1}{2}} X \exp(-X^2) \tag{48}$$

Also we note that $F_i(X) = F_e(\kappa X)$.

In Eq. (45) the quantity $\pi^{\frac{1}{2}} X$ appears as a factor and is taken outside the fraction and combined with the first factor. Dividing Eq. (45) by the resulting new first factor we obtain

$$H(X) = \frac{kV}{n_0} \left(\frac{2\pi KT}{m}\right)^{\frac{1}{2}} <|n(\vec{k},\omega)|^2>_{av} = \frac{\exp(-X^2)|1+2X_p^2 F_i|^2 + 4X_p^4 \kappa \exp(-\kappa^2 X^2)|F_e|^2}{|1+2X_p^2(F_e + F_i)|^2}$$

$$\tag{49}$$

The right-hand side of this was computed on an electronic computer for values of X_p ranging from 300 to 0.03 for $\kappa = 43$ and $\kappa = 172$. The results are shown in Fig. 2.

Let us try to interpret these curves in as simple physical terms as possible. Consider density variations of one scale k^{-1} only. The time variations of the density fluctuation at this scale may be thought of as a superposition of highly damped, plane longitudinal electron waves with a wide range of different phase velocities. These waves are excited through interaction with the microscopic motion in the plasma. For large scales k^{-1} the electrons are tied to the ion motion and the spectrum is identical to that of the ion motion. This case is represented in Fig. 2 by the curve for $X_p = 300$. As the scale k^{-1} is decreasing and becomes of the order of the Debye length the electron motion is no longer completely tied to the motion of the ions. In fact, the fast electron waves are actually becoming independent of the ion motion before the slow ones. This explains why, in the transition region, the spectral density increases with velocity in the velocity range
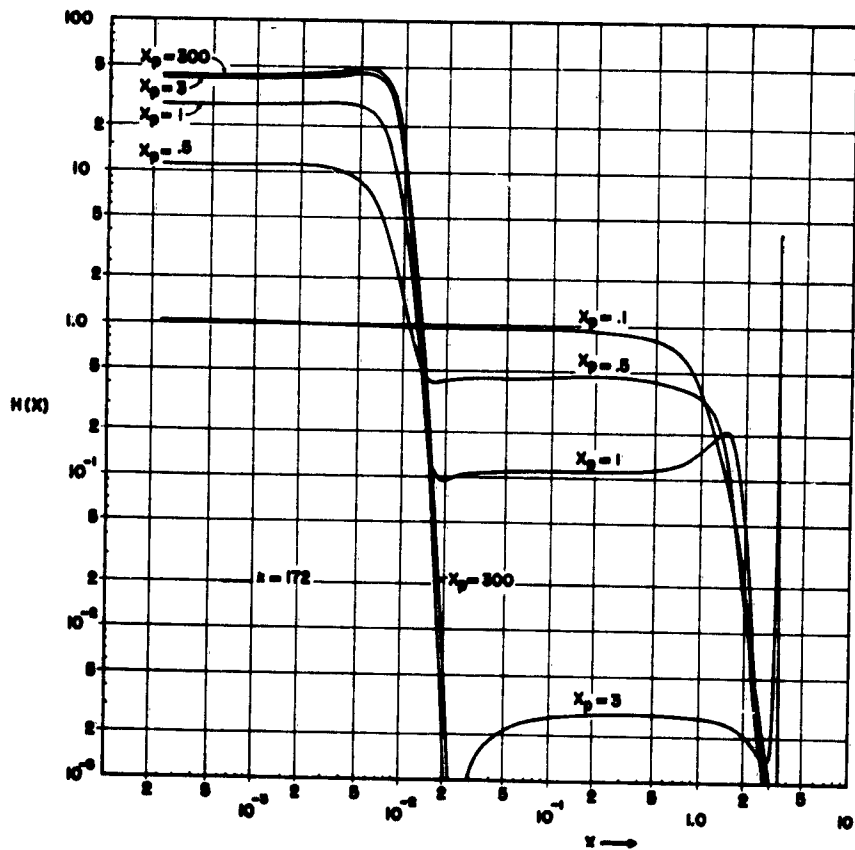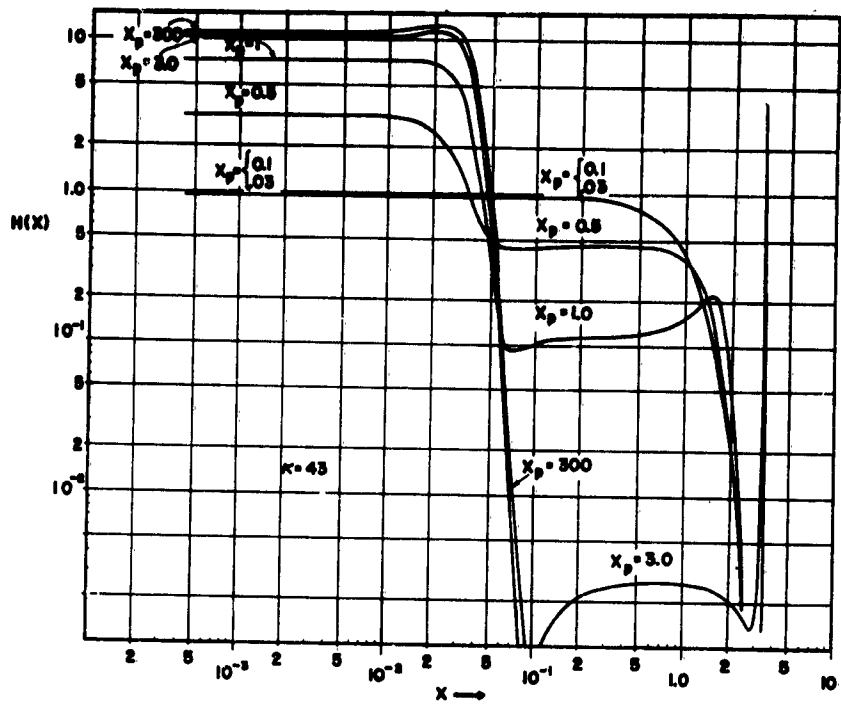
FIG. 2. $H(X) = \dfrac{kV}{n_o} \left(\dfrac{2\pi KT}{m}\right)^{\frac{1}{2}} < |n(\vec{k},\omega)|^2 >_{av}$ PLOTTED AGAINST X FOR THE LONGITUDINAL CASE.

where the fast electron waves are able to move independently of the ion motion. In particular, the pronounced dip in the curves for $X_p = 1.0$ and 3.0 can be explained this way. When X_p becomes small the spectrum tends toward a Gaussian, as it should when the electrons become completely free to move.

For large X_p there is a very sharp peak in the curves for $X \approx X_p$. This corresponds to the familiar electrostatic waves at the plasma frequency. The contribution to the total fluctuation from this peak is, in general, negligible.

It should also be noted that the widths of the curves for large X_p and for small X_p are in the ratio κ^{-1}.

5.3 GENERAL CASE

Useful approximate expressions for F_e and F_i may be developed in the general case of $\theta \neq 0$. Let us assume that collisions are unimportant and that we avoid the case of strictly transverse propagation. Then the integral in F_e and F_i can be solved by expanding

$$\exp\left(\frac{\sin^2\theta}{2X_e^2}\cos y\right)$$

in a Fourier series and integrating term by term.

The expressions for the real and imaginary parts of F_e then become

$$\text{Re}(F_e) = 1 - \frac{2X}{\cos\theta}\exp\left(-\frac{\sin^2\theta}{2X_e^2}\right)\sum_{n=-\infty}^{+\infty} I_n\left(\frac{\sin^2\theta}{2X_e^2}\right)\exp\left[-\left(\frac{1}{\cos\theta}\right)^2(X-nX_e)^2\right]$$

$$\int_0^{\frac{1}{\cos\theta}(X-nX_e)} e^{\phi^2}\,d\phi$$

$$\text{Im}(-F_e) = \pi^{\frac{1}{2}}\frac{X}{\cos\theta}\exp\left(-\frac{\sin^2\theta}{2X_e^2}\right)\sum_{n=-\infty}^{+\infty} I_n\left(\frac{\sin^2\theta}{2X_e^2}\right)\exp\left[-\left(\frac{1}{\cos\theta}\right)^2(X-nX_e)^2\right]$$

(50)

where I_n is the Bessel function of imaginary argument of order n.

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Because these sums are rather difficult to handle we shall here
deal only with approximations for large and small radii of gyration.
In ionospheric applications, the former case applies to the ions and
the latter to the electrons, for a fairly large range of wave vectors $\vec{k}$.

When the radius of gyration is small, $X_e > 1$ and only the zeroth
order terms in the sums above need be considered as long as X_e also
exceeds X. Hence:

$$\text{Re}(F_e) \cong 1 - \frac{1}{\cos\theta} \exp\left[-\left(\frac{X}{\cos\theta}\right)^2\right] \int_0^{\frac{X}{\cos\theta}} e^{\emptyset^2}\, d\emptyset$$

$$\text{Im}(-F_e) \cong \pi^{\frac{1}{2}} \frac{X}{\cos\theta} \exp\left[-\left(\frac{X}{\cos\theta}\right)^2\right] \qquad (51)$$

Note that these equations could have been obtained from the longitudinal
case by replacing X by $X/\cos\theta$.

When the radius of gyration is large an approximation can best
be obtained from the original expression for F_e rather than from
Eqs. (50). In this case $X_e < 1$, and the expression

$$\exp\left[-\frac{\sin^2\theta}{2X_e^2} (1 - \cos y)\right]$$

will be appreciably different from zero only when $y = 2\pi n$, provided
$\sin\theta > X_e$. The contributions around $y = 2\pi n$ are obtained by expand-
ing $1 - \cos y$ to second order, i.e., by putting

$$1 - \cos y \approx \frac{1}{2} (y - 2\pi n)^2$$

The expressions for the real and imaginary parts of F, specialized
for the ions through the substitution (see p. 16)

$$X \rightarrow \kappa X$$
$$X_e \rightarrow \kappa^{-1} X_e$$

become

$$\mathrm{Re}(F_i) \overset{\sim}{=} 1 - 2 \kappa X \exp\left[-\kappa^2 X^2\right] \left\{ \int_0^{\kappa X} e^{\phi^2} d\phi + \pi^{\frac{1}{2}} \sum_{m=1}^{\infty} \exp\left[-\left(\frac{\pi m \kappa}{X_e}\right)^2 \sin^2\theta \cos^2\theta\right] \right.$$

$$\left. \sin\left[2\pi m \kappa^2 \sin^2\theta \frac{X}{X_e}\right] \right\}$$

$$\mathrm{Im}(-F_i) \overset{\sim}{=} \pi^{\frac{1}{2}} \kappa X \exp\left[-\kappa^2 X^2\right] \left\{ 1 + 2 \sum_{m=1}^{\infty} \exp\left[-\left(\frac{\pi m \kappa}{X_e}\right)^2 \sin^2\theta \cos^2\theta\right] \right.$$

$$\left. \cos\left(2\pi m \kappa^2 \sin^2\theta \frac{X}{X_e}\right) \right\} \qquad (52)$$

The effect of the magnetic field is to modulate the functions obtained in the longitudinal case. It is only when θ is fairly close to $90°$ that that this modulation becomes appreciable.

Because of the number of parameters involved it is not possible to compute sets of curves valid for as many different cases as for the nonmagnetic case of Fig. 2. In order to discuss the effect of the magnetic field on the spectral distribution, we therefore choose one particular set of parameters and study the change of the spectrum with angle θ.

We choose parameters:

$$X_e = 10$$
$$X_p = 30$$
$$\kappa = 43 \quad \text{(hydrogen)}$$

Let us discuss the significance of this choice. As mentioned above, the electron gyro-frequency is taken as 1 Mc/s. Because $X_p/X_e = 3$ this means that the plasma frequency is 3 Mc/s, which is at least of the right order of magnitude above the peak of the F-layer of the ionosphere. If the temperature is $1000°$K, then $\left(\frac{m}{2KT}\right)^{\frac{1}{2}} = 5.7 \cdot 10^{-6}$. With $X_e = 10$ in a backscatter experiment the radio wavelength would correspond to 3.5 meters, which is typical of the wavelengths used at present.
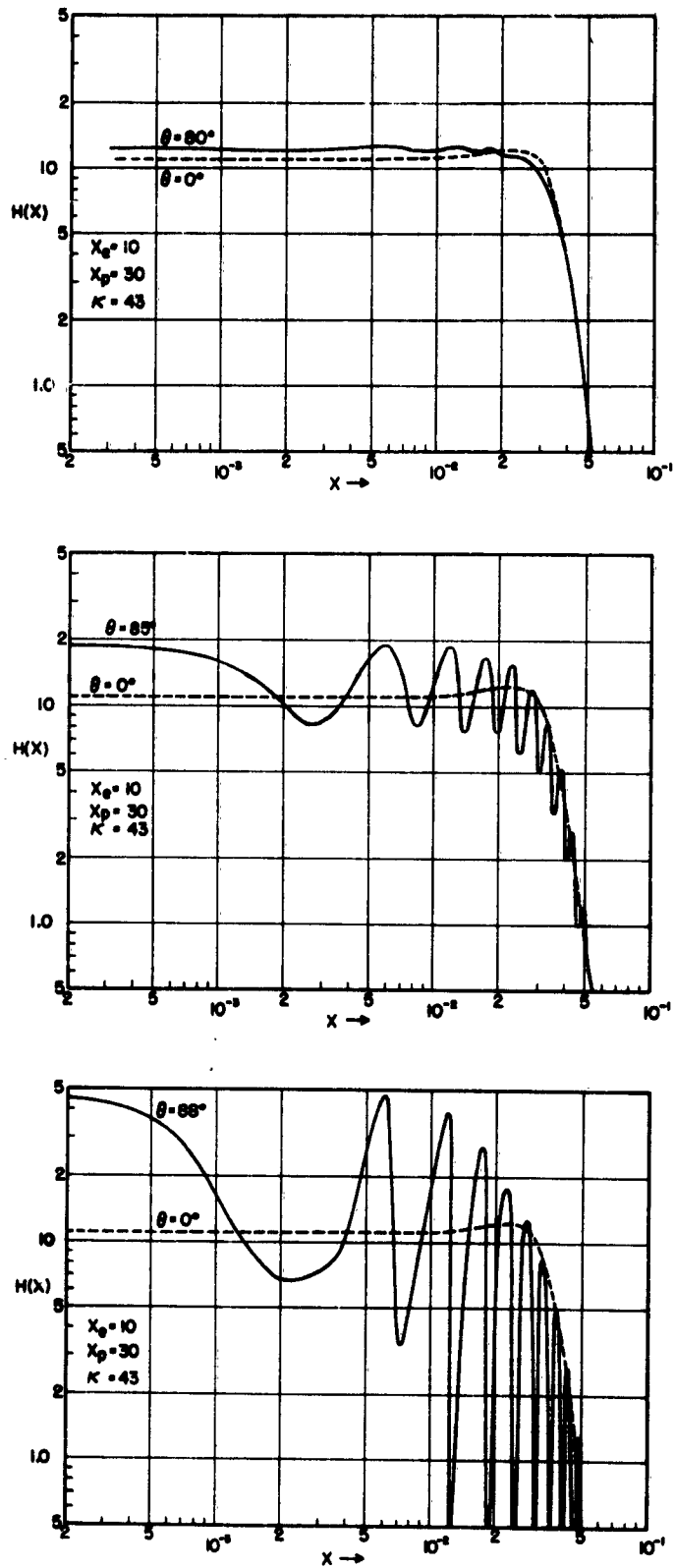
FIG. 3.   EXAMPLE OF THE CHANGE IN H(X) AS θ TENDS TOWARDS 90°.

Substituting Eqs. (51) and (52) into Eq. (45) and normalizing as in the longitudinal case (Eq. 49) we can again compute

$$H(X) = \frac{kV}{n_0} \left(\frac{2\pi KT}{m}\right)^{\frac{1}{2}} <|n(\vec{k},\omega)|^2>_{av}$$

against X on an electronic computer. The computation was carried through for $\theta = 80°$, $85°$ and $88°$, and the "ionic" parts of the spectra are shown in Fig. 3. The longitudinal case for $X_p = 30$ is shown dotted.

We see that as θ approaches $90°$ the spectrum develops toward a line spectrum with peaks at the gyro-frequencies of the ions, and peaks at all the harmonics of this frequency. Because of the computer time required the range of the power density was limited. The frequency range was also reduced, for the same reason, to the range of X where the spectrum is essentially of ionic nature.

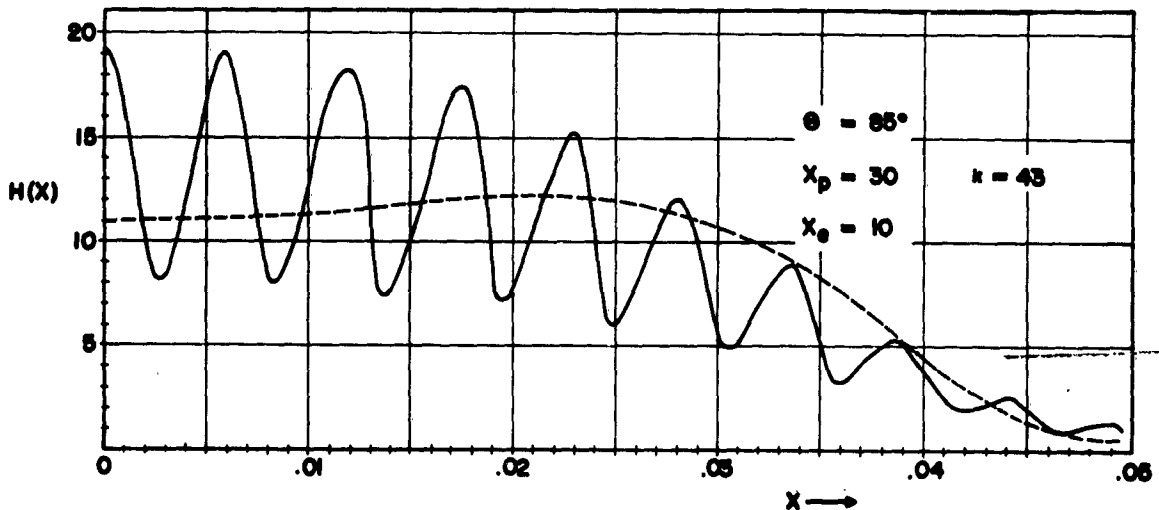The spectrum for $\theta = 85°$ is displayed on a linear-linear scale in Fig. 4.

FIG. 4. PLOT OF H(X) AGAINST X ON A LINEAR SCALE.

VI. DISCUSSION AND CONCLUSION

The present work has shown how the spatial Fourier components of electron density fluctuations and their spectral distribution may be computed for a plasma in thermal equilibrium in an external magnetic field when Coulomb interaction is taken into account.

In the ionosphere we will be concerned with only $Z = 1$. Thus, if the "scale" k^{-1} is larger than the Debye length, the total fluctuation is found to be half that found in the absence of interactions. If the "scale" is smaller than the Debye length, the total fluctuation is not influenced by interactions. The total fluctuation, and hence the total scattered power, is independent of the presence of a static magnetic field.

For physical conditions which are believed to prevail in the ionosphere, and for wavelengths (scales) of the order of a few meters, the spectral distribution is not influenced by the presence of a magnetic field unless the wave vector $\vec{k}$ is nearly perpendicular to the magnetic field. As $\vec{k}$ approaches perpendicularity, sharp resonance peaks are developing rapidly round the ionic gyro-frequency and multiples thereof.

It therefore appears that it will be possible to study the ionic constituents in the ionosphere by carefully designed radio-wave-scatter experiments. The temperature can also be found by studying the spectral width, either of the "ionic" spectrum or of the "electronic" spectrum. The electron density may be found from the total scattered power.

The possibility of actually detecting the peaks at the ionic gyro-frequencies depends on the frequency resolution available. In normal backscatter observations with relatively short pulses there is a chance that the details may be washed out because of the width of the spectrum of the transmitted wave. It therefore appears that some sort of bistatic continuous-wave experiment ought to be looked into and thoroughly discussed.

When several ionic components are present the above theory can be extended without any difficulties in principle, but the work becomes more laborious and is omitted here.

ACKNOWLEDGMENT

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APPENDIX A

The solution of the homogeneous part of Eq. (30) is:

$$C_0(w,u) = \exp\left\{\frac{1}{\mu B}\left[(s - iku \cos \theta)\emptyset - ikw \sin \theta \sin \emptyset\right]\right\} \quad (A1)$$

Instead of $C_0(w,u)$ we substitute a function $C(w,u,\emptyset)$ and try to solve the inhomogeneous equation. It is then found that:

$$C(w,u,\emptyset) = \frac{1}{\mu B}\int_{\substack{\text{fixed} \\ \text{limit}}}^{\emptyset} \exp\left\{-\frac{1}{\mu B}\left[(s - iku \cos \theta)\emptyset' - ikw \sin \theta \sin \emptyset'\right]\right\}$$

$$\left\{\frac{\mu}{n_0}\frac{\partial n_0'}{\partial \vec{v}'}\vec{E} - n^1(\vec{k},\vec{v}')\right\} d\emptyset' \quad (A2)$$

The full solution therefore becomes:

$$n_1(w,u,\emptyset) = \frac{1}{\mu B}\int_{\substack{\text{fixed} \\ \text{limit}}}^{\emptyset} \exp\left\{\frac{1}{\mu B}\left[(s - iku \cos \theta)(\emptyset - \emptyset') - ikw \sin \theta\right.\right.$$

$$\left.\left.(\sin \emptyset - \sin \emptyset')\right]\right\}\left\{\frac{\mu}{n_0}\frac{\partial n_0'}{\partial \vec{v}'}\vec{E}(\vec{k},s) - n'(\vec{k},\vec{v}')\right\} d\emptyset' \quad (A3)$$

where $\vec{v}'$ is obtained from $\vec{v}$ by putting $\emptyset = \emptyset'$. Because physical considerations require that the function be single-valued, we must construct a solution which is periodic in $\emptyset$. The only possible solution is then the one presented in Eq. (31).

From Eqs. (32), (33) and (35) we see that:

$$Y_e = - \int_{\vec{v}} \int_{-\infty}^{\emptyset} G_e(\emptyset,\emptyset')n_o(\vec{v})n_1(\vec{k},\vec{v}')d(\vec{v})d\emptyset'$$

(B1)

$$R_e = - \int_{\vec{v}} \int_{-\infty}^{\emptyset} \vec{kv}'G_e(\emptyset,\emptyset')n_o(\vec{v})d(\vec{v})d\emptyset'$$

and:

$$Y_i = - \int_{\vec{V}} \int_{\infty}^{\emptyset} G_i(\emptyset,\emptyset')N_o(\vec{V})N_1(\vec{k},\vec{V}')d(\vec{V})d\emptyset'$$

(B2)

$$R_i = - \int_{\vec{V}} \int_{\infty}^{\emptyset} \vec{kV}'G_i(\emptyset,\emptyset')N_o(\vec{V})d(\vec{V})d\emptyset'$$

The expressions for R may be integrated immediately, but not so those for Y because of the stochastic nature of $n^1(\vec{k},\vec{v}')$ and $N^1(\vec{k},\vec{V}')$. Before integrating we must form the required averages $\langle n*(\vec{k})Y_e \rangle_{av}$ and $\langle n*(\vec{k})Y_i \rangle_{av}$ and make use of Eqs. (44). The following results are then obtained:

$$\langle n*(\vec{k})Y_e \rangle_{av} = - \frac{1}{V} \frac{1 + 2X_p^2 Z}{1 + 2X_p^2(1 + Z)} \int_{\vec{v}} \int_{-\infty}^{\emptyset} G_e(\emptyset,\emptyset')n_o(\vec{v})d(\vec{v})d\emptyset'$$

(B3)

$$\langle n*(\vec{k})Y_i \rangle_{av} = - \frac{1}{V} \frac{2X_p^2 Z}{1 + 2X_p^2(1 + Z)} \int_{\vec{V}} \int_{\infty}^{\emptyset} G_i(\emptyset,\emptyset')N_o(\vec{V})d(\vec{V})d\emptyset'$$

(B4)

Bernstein (1958) has shown how this type of integral can be evaluated. Because the procedure is fairly straight forward the details are omitted

here. The integral in (B3) becomes:

$$\frac{n_o}{\Omega_e} \int_o^\infty dy \exp\left\{ -\left(\frac{s}{\Omega_e}\right)y - [\sin^2\theta(1 - \cos y) + \tfrac{1}{2}y^2\cos^2\theta]\frac{KT\kappa^2}{m\Omega_e^2} \right\}$$

The integral in (B4) is of exactly the same form, we have only to substitute parameters applying to ions instead of electrons. The integral, which we will term a Gordeyev integral, will now be denoted by $g(\theta, s/\Omega)$, and we put indices e or i to indicate whether it applies to electrons or ions.

In terms of this integral the expressions (B3) and (B4) become:

$$\langle n*(\vec{k})Y_e\rangle_{av} = \frac{n_o}{\Omega_e V}\frac{1 + 2X_p^2 Z}{1 + 2X_p^2(1 + Z)}\; g_e\left(\theta, \frac{s}{\Omega_e}\right) \qquad (B5)$$

$$\langle n*(\vec{k})Y_i\rangle_{av} = \frac{n_o}{\Omega_i V}\frac{1 + 2X_p^2 Z}{1 + 2X_p^2(1 + Z)}\; g_i\left(\theta, \frac{s}{\Omega_i}\right) \qquad (B6)$$

The integrals occurring in R_e and R_i may also be expressed in terms of the Gordeyev integral:

$$R_e = in_o\left[1 - \frac{s}{\Omega_e}\; g_e\left(\theta, \frac{s}{\Omega_e}\right)\right] = in_o F_e\left(\theta, \frac{s}{\Omega_e}\right) \qquad (B7)$$

$$R_i = i\frac{n_o}{Z}\left[1 - \frac{s}{\Omega_i}\; g_i\left(\theta, \frac{s}{\Omega_i}\right)\right] = i\frac{n_o}{Z} F_i\left(\theta, \frac{s}{\Omega_i}\right) \qquad (B8)$$

Our final formulae (45), (46) and (47) follow from these by simple substitution.